Constructible circles on the unit sphere

Blaga Slavcheva Pauley

CONSTRUCTIBLE CIRCLES ON THE UNIT SPHERE

A Project

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

---

by

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ABSTRACT

In this paper we show how to give an intrinsic definition of a constructible circle on the sphere. The classical definition of constructible circle in the plane, using straight edge and compass is there by translated in terms of so called Lenart tools. The process by which we achieve our goal involves concepts from the algebra of Hermitian matrices, complex variables, and Stereographic projection. However, the discussion is entirely elementary throughout and hopefully can serve as a guide for teachers in advanced geometry.

ACKNOWLEDGMENTS

I want to express my gratitude to my incredible mentors, Dr. John Sarli and Dr. Susan Addington. Dr. Robert Stein filled my days with humor and hidden pictures. I will always treasure his genuine acceptance of who I am, and his contagious love for the history of mathematics. To be honest, I never intended to test the limits of Dr. Hasan's patience in the classroom. It was a case of both student and teacher being equally passionate about the subject. I am grateful to Dr. Williams and Dr. Hallet for the emotional support through my first clash with a "difficult student." I honor all of my beloved teachers in the last four incredible years - Dr. Griffing, Dr. Freiling, Dr. Joel Stein, Dr. Stanton, Dr. Rinne, Dr. Wang, and Dr. Fejzic. I am grateful to Dr. Davida Fishman for providing her warm friendship and good advice, when I needed it. My love goes to Paula, Nancy and Heather- the Three Graces of the Department; to my best friend Steven; and to the rest of my colleagues- the extraordinary graduate students of Class 2000. Above all, I honor and admire the people who lovingly and patiently, let me realize all of my dreams- my husband Tom, my three precious boys, and Mom.

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I. INTRODUCTION

One of the oldest games in mathematics is geometric construction. As specified by Plato, the tools of the game are the ruler (straight edge) and the compass. Points, lines, and circles on the plane, constructed by these tools only, are called "ruler and compass" points, lines, and circles, or simply "constructible" points, lines, and circles.

In the Cartesian plane, a constructible point is a point which is the last of a finite sequence $P_1, P_2, \ldots, P_n$ of points such that each point is in $\{(0,0), (1,0)\}$ or is obtained in one of three ways: (i) as an intersection of two lines, each of which passes through two points that appear earlier in the sequence; (ii) as an intersection of a line and a circle, where the line passes through two points that appear earlier in the sequence and the circle passes through an earlier point, having another earlier point as a center; and (iii) as an intersection of two circles, each of which passes through an earlier point in the sequence and has an earlier point as a center.

A *constructible line* is a line that passes through two constructible points.

A *constructible circle* is a circle through a constructible point and having another constructible point as a center, or equivalently, a circle with a constructible point as a center and having a constructible radius.

A number x is a *constructible number* if (x, 0) is a constructible point. [1]

With the definitions above, we have an Euclidean notion for the constructible circles and constructible lines. Yet, on the surface, the definitions of constructible lines and circles appear to be different. A common definition can be found through stereographic projection of the constructible circles in the plane onto the unit sphere.

The main idea in this paper is the intrinsic characterization of the constructible circles on the unit sphere.

In Section II the circles of Apollonius will be defined in the realm of the Euclidean plane as the hyperbolic pencil of circles. The definition will be derived in the context of two other pencils of circles, in particular, the elliptic and the parabolic pencils of circles.

In Section III, a new representation of circles on the plane will be introduced, namely the Hermitian matrix representation of circles on the plane.

In Section IV, the tool of the stereographic projection will be defined and used to transmit the pencils of circles on the Euclidean plane onto the unit sphere.

Finally, in Section V, the intrinsic definition of the constructible circles on the unit sphere will be derived.

II.CIRCLES OF APOLLONIUS

We will center our discussion on the ancient problem of Apollonius, but before we give that problem, we will set out a few definitions.

(2.1) DEFINITION:

Reflection σ_m *in line* m is an involutory transformation that interchanges the half planes of m.

Reflection σ_m fixes point P if and only if P is on m.

Reflection σ_m fixes line l pointwise if and only if $l=m$.

Reflection σ_m fixes line l if and only if $l=m$ or $l \perp m$. [6]

(2.2) DEFINITION:

Given a circle with center O and radius k, the *inverse* of any point $P \neq O$ is $P' \in \overrightarrow{OP}$, whose distance from O satisfies the equation $OP \times OP' = k^2$. [4]

It follows from the definition 2.2 that the inverse of P' is P.

Inversion has two very important properties. It is an involutory and a conformal transformation. The later is a transformation, which preserves angles.

Inversion can be seen as a "reflection" in a circle [W. Blaschke, *Analytische Geometrie,* p. 47]. Such an analogy allows us to define a straight line as a circle of infinite radius. From that point of view, any circle inverts into a circle.

The transformation defined above as *inversion* was characterized in 1828 by Jacob Steiner (1796-1863), a mathematician from Switzerland, who has been regarded as the greatest synthetic geometer since Euclid. Remarkably, Jacob Steiner did not publish his ideas on inversive geometry and William Thompson Lord Kelvin (1824-1907) discovered them independently through physics. The ideas of inversive geometry found their applications in problems in electrostatics.

To familiarize ourselves with inversion we may follow a simple example. We will discover what kind of curve is the inversion of the rectangular (or "equilateral") hyperbola in the unit circle.

<u>*(2.3) DEFINITION:*</u>

Let $z=(x,y)\in C \ni z=x+iy$. The *inversion* of z is $\frac{1}{z}=\frac{|z|}{|z|^2}$

<u>*(2.4) DEFINITION:*</u>

A *hyperbola* is the set of all points on the plane, the difference of whose distances from two fixed points is a given positive constant that is less than the distance between the fixed points. When centered at the origin, the equations of the hyperbola are:

1. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if foci are on the x-axis. When $a=b=1$ then the unit hyperbola with foci on the x-axis is

$$x^2-y^2=1$$

2. $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ if foci are on the y-axis. When $a=b=1$ then the unit hyperbola with foci on the y-axis is

$$y^2-x^2=1 \quad [7]$$

Our investigation will take place on the complex plane, with points of the form $(x,y)=z$ such that $z=x+iy$. The *inversion of* z is defined by $\frac{1}{z}$. Then the points on the inversion of the unit hyperbola in the unit circle are of the form $\frac{1}{x-iy}$.

It follows that $\frac{1}{z}=\frac{1}{x-iy}\times\frac{x+iy}{x+iy}=\frac{x}{x^2+y^2}+\frac{y}{x^2+y^2}i$, which corresponds to the points on the complex plane of the form

$$\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right).$$

By definition, for the points (x,y) the equation of unit hyperbola with foci on the x-axis is of the form $x^2 - y^2 = 1$. What curve do the points $\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ form if $x^2 - y^2 = 1$?

Now, $\left(\frac{x}{x^2+y^2}\right)^2 - \left(\frac{y}{x^2+y^2}\right)^2 = 1$ is a curve of order 4. Let's translate it into polar coordinates. (Recall that the conversion from rectangular to polar coordinates is defined by the equations $x = r\cos\theta$ and $y = r\sin\theta$.)

$$\left(\frac{x}{x^2+y^2}\right)^2 - \left(\frac{y}{x^2+y^2}\right)^2 = 1$$

$$\frac{r^2\cos^2\theta}{r^4} - \frac{r^2\sin^2\theta}{r^4} = 1$$

$$\cos^2\theta - \sin^2\theta = r^2$$

$$\cos\ 2\theta = r^2 \tag{2.a}$$

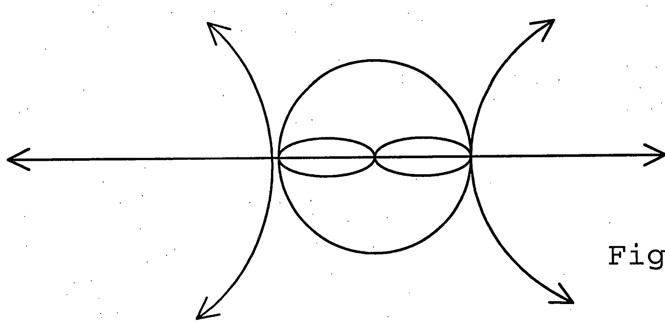


Figure 2.1

(2.5) DEFINITION:

If $a > 0$, then the equations of the form

$$r^2 = a^2 \cos 2\theta \quad r^2 = -a^2 \cos 2\theta$$

$$r^2 = a^2 \sin 2\theta \quad r^2 = -a^2 \sin 2\theta$$

represent propeller-shaped curves, called *lemniscates* (from Greek word "lemniscos" for a looped ribbon resembling the figure 8). The lemniscates are centered at the origin, but the position relative to the polar axis depends on the sign preceding the a^2 and whether $\sin 2\theta$ or $\cos 2\theta$ appears in the equation. [7]

For a=1, we recognize equation (2.a) as one of the four standard equations for a *lemniscate*.

Therefore, the inversion of the unit hyperbola with vertex at the origin, in the unit circle centered at the origin is the lemniscate on the x-axis, centered at the origin, also known as *the lemniscate of Bernoulli*.

In general, a circle $x^2 + y^2 = r^2$ centered at the origin, with radius r inverts any point (x, y) on the Euclidean plane into

$$\left(\frac{r^2 x}{x^2 + y^2}, \frac{r^2 y}{x^2 + y^2} \right)$$

A more general definition of the inversion in an arbitrary circle on the complex plane is given by

(2.6) THEOREM:

An inversion in a circle C of radius r, centered at (a,b) may be represented in the complex plane by the transformation

$$t(z)=\frac{r^2}{z-c}+c\text{, where } (z\in C-\{c\}) \qquad (2.b)$$

As another example of inversion, we will derive the formulae for inversion of an arbitrary circle in the unit circle centered at (1,0). Without loss of generality, we may assume $a=1$ and radius $\rho=1$. The point circle then is at (1,0).

$$c=a+bi=1+0i=1$$

$$z-c=(x+iy)-1=(x-1)+iy$$

$$\overline{z-c}=(x-1)-iy$$

$$r=1=r^2$$

In the context of the hyperbolic and elliptic pencils (to be defined below), the inversion in a unit circle centered at (1,0) will be

$$t(z)=\frac{1}{(x-1)-iy}+1$$

which after rationalizing the denominator simplifies to

$$\frac{(x-1)^2+(x-1)+y^2}{(x-1)^2+y^2}+\frac{y}{(x-1)^2+y^2}i$$

Therefore, the image of any point (x,y) on a circle in the elliptic or hyperbolic pencils will be a point on the complex plane defined as

$$\left(\frac{(x-1)^2+(x-1)+y^2}{(x-1)^2+y^2},\frac{y}{(x-1)^2+y^2}\right)$$

The general equation of an arbitrary circle in the elliptic pencil is

$$x^2+y^2-2ky-1=0 \qquad (2.c)$$

Therefore, the inversion of an arbitrary circle through the unit circle centered at (1,0) would be

$$\left(\frac{x^2-x+y^2}{(x-1)^2+y^2}\right)^2+\left(\frac{y}{(x-1)^2+y^2}\right)^2-2k\left(\frac{y}{(x-1)^2+y^2}\right)=1$$

$$\frac{x^2+y^2-2ky}{x^2+y^2-2x+1}=1$$

which yields to

$$x^2+y^2-2ky=x^2+y^2-2x+1$$

$$2x - 2ky = 1 \qquad (2.d)$$

Similarly, the general equation of an arbitrary circle in the hyperbolic pencil is

$$x^2 + y^2 + 2ky - 1 = 0 \qquad (2.e)$$

$$\left(\frac{x^2 - x + y^2}{(x-1)^2 + y^2}\right)^2 + \left(\frac{y}{(x-1)^2 + y^2}\right)^2 + 2k\left(\frac{y}{(x-1)^2 + y^2}\right) = 1$$

$$\frac{x^2 + y^2 + 2ky}{x^2 + y^2 - 2x + 1} = 1$$

which yields to

$$x^2 + y^2 + 2ky = x^2 + y^2 - 2x + 1$$

$$2x + 2ky = 1 \qquad (2.f)$$

We will now give an elementary definition of pencil of circles. Any single circle in the Euclidean plane may belong to any of three distinct pencils. Any two circles on the Euclidean planes define a particular pencil of circles.

(2.7) DEFINITION:

A collection of circles determined by two distinct circles is called a *pencil of circles.* Two distinct circles may relate in three ways only:

a) they may intersect at two points *(elliptic pencil);*

b) they may be tangent to each other *(parabolic pencil)*; and

c) they may fail to intersect *(hyperbolic pencil of circles.)*

(2.8) DEFINITION:

An *elliptic (intersecting) pencil* of circles consists of all the circles passing through two fixed points. The elliptic pencil of circles has no point circles. There is a unique line circle through P_1 and P_2.

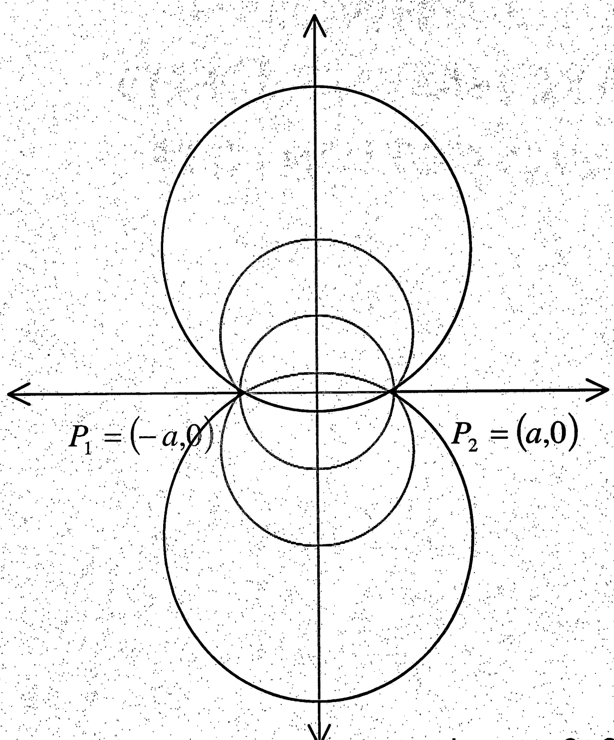


Figure 2.2

We may take the line circle through P_1 and P_2 as the $x-axis$ of our coordinate system. The line circle defined by all the centers of the circles in the pencil passing through the fixed points P_1 and P_2, is the perpendicular bisector of $\overline{P_1P_2}$. We take it as the $y-axis$ of the coordinate system we need to work with. Therefore, the center of an arbitrary circle in the pencil would have the coordinate of the form $(0,k)$. Since the ordinate is the perpendicular bisector, the fixed points P_1 and P_2 will be equidistant from the origin and the coordinates for P_1 and P_2 should be of the form $(-a,0)$ and $(a,0)$, respectively. (Note that in this context, the $y-axis$ appears to be the line of reflection for the fixed points P_1 and P_2.)

(2.9) DEFINITION:

The *parabolic (tangent) pencil* contains all the circles sharing the point of tangency but also whose centers lay on the line determined by the centers of the two original circles. The pencil has exactly one real *point circle*, situated at the only common point of all the circles of the pencil. All circles of the pencil are real.

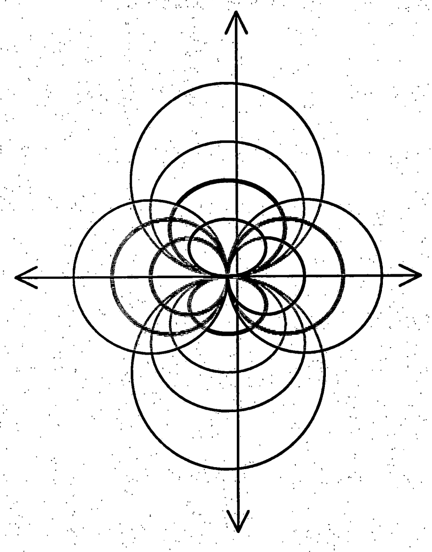


Figure 2.3

Let P be an arbitrary point on a vertical line l. Draw all the circles with centers on l passing through the point P. We will easily reason that the "*unique line circle*" through P will be a straight line, perpendicular to l, therefore a horizontal line m, such that $l \cap m = P$. If we assign l and m as $y-$ and $x-axes$ respectively, now we have a rectangular coordinate system we can work with.

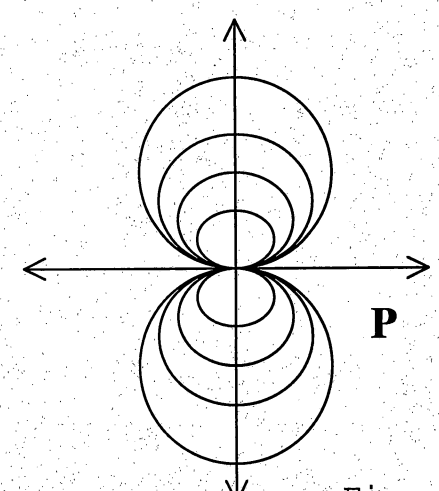


Figure 2.4

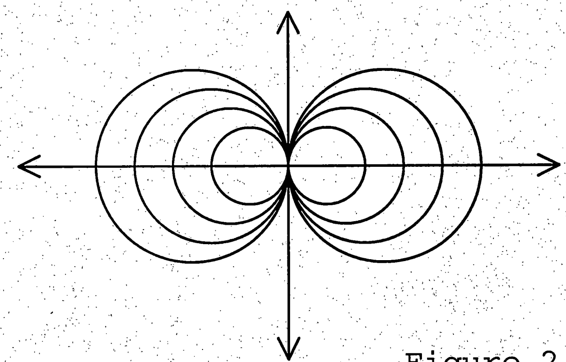


Figure 2.5

Without loss of generality, the lines l and m could interchange roles. If m is seen as the line formed by the points, which are the centers of circles passing through the point P, then the line l would be the *"unique line circle."* In this case the *parabolic pencil of circles* will be on the $x-axis$ of our rectangular coordinate system.

The proof of the orthogonality of the set of all circles with their centers on the y-axis and the set of all circles with their centers on the x-axis will occur in Section III, after the Hermitian matrix representation of the pencils of circles is introduced.

An important distinction between parabolic and elliptic pencils of circles that a parabolic pencil is determined by a unique point only, while an elliptic pencil of circles is determined by both, the points P_1 and P_2, and also by their distance $d(P_1, P_2)$ from each other. Another important observation is that there exists a *minimal radius* for the circles in a parabolic pencil, which is exactly

$$\frac{d(P_1, P_2)}{2}.$$

Given that $P_1 = (-a, 0)$, $P_2 = (a, 0)$, and the center of an arbitrary circle $O = (0, k)$, we may find the general expression for the radii of the circles in the elliptic pencil.

$$(x-h)^2 + (y-k)^2 = \rho^2$$

$$(x-0)^2 + (y-k)^2 = \rho^2$$

$$x^2 + (y-k)^2 = \rho^2$$

$$\sqrt{x^2 + (y-k)^2} = \rho \tag{2.g}$$

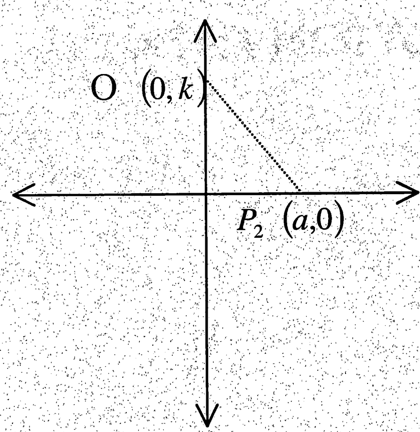


Figure 2.6

Also, using the Distance Formula, we may express the radius of an arbitrary

$$\sqrt{(0-a)^2+(k-0)^2}=\rho$$

$$\sqrt{a^2+k^2}=\rho$$

Combining both results we have the equation

$$\sqrt{x^2+(y-k)^2}=\sqrt{a^2+k^2}$$

$$x^2+(y-k)^2=a^2+k^2 \qquad (2.h)$$

The third case in the definition of pencil of circles is really the starting point of our discussion. How does one determine a collection of circles by two nonintersecting circles? A rather surprising answer of this question will bring us back to an ancient Greek problem, stated by Apollonius of Perga (c 260-190 BC).

Now we will introduce the construction of the circle of Apollonius as a way of completing the classification of pencils of circles.

(2.10) THEOREM:

The *locus* of a point P whose distances from two fixed points P_1 and P_2 are in a constant ratio $1:\mu$, so that

$\overline{P_2P} = \mu\overline{P_1P}$ is a circle.

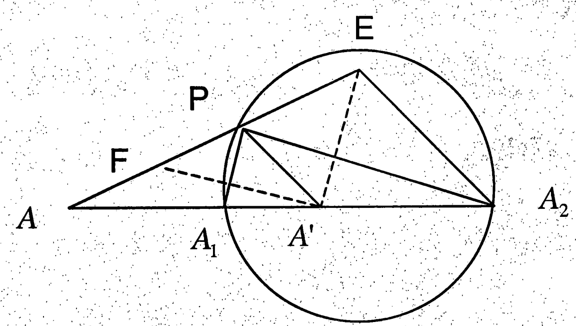


Figure 2.7

PROOF: Assuming $\mu \neq 1$ let P be any point for which $A'P = \mu AP$. (Refer to Figure 2.7 on the previous page.) Let the internal bisector of $\angle APA'$ meet with AA' at point A_1. Let the external bisector of $\angle APA'$ meet AA' at point A_2. Construct E and F on AP such that $A'E$ is parallel to A_1P and $A'F$ is parallel to A_2P, which makes it perpendicular to A_1P. Now $FP = PA' = PE$, then it follows that

$$\frac{AA_1}{A_1A'} = \frac{AP}{PE} = \frac{AP}{PA'} \quad \text{and also} \quad \frac{AA_2}{A'A_2} = \frac{AP}{FP} = \frac{AP}{PA'}.$$

The points A_1 and A_2 divide the segment $\overline{AA'}$ in the ratio $1:\mu$ and moreover, their location is independent of the position of the point P. Since $\angle A_1PA_2$ is a right angle, *P lies on the circle with diameter* A_1A_2.

Conversely, let the points A_1 and A_2 be defined as the points dividing the line segment $\overline{AA'}$ in the ratio $1:\mu$. Let the point P be any point on the circle with diameter A_1A_2. Then:

$$\frac{AP}{PE}=\frac{AA_1}{A_1A'}=\frac{1}{\mu}=\frac{AA_2}{A'A_2}=\frac{AP}{FP}$$

Thus $FP=PE$, and P, which is the midpoint of FE, is the circumcenter of the right triangle $\angle EA'F$. It follows that $PA'=PE$ and

$$\frac{AP}{PA'}=\frac{AP}{PE}=\frac{1}{\mu}$$

The most important discovery is that A' is an *inversion* of A through the *circle of Apollonius* A_1A_2P. Let's investigate the relations between the lengths of the line segments formed by the points A, A', A_1, A_2, and the center of the circle O.

Let $\overline{A_1O}=\overline{A_2O}=\rho$ the radius of the circle A_1A_2P. Observe that

$$\overline{AA_1}=\overline{AO}-\rho$$

$$\overline{A_1A'}=\rho-\overline{A'O}$$

$$\overline{AA_2}=\rho+\overline{AO}$$

$$\overline{A_2A'}=\rho+\overline{A'O}$$

Now we may derive the proportions

$$\frac{\overline{A0}-\rho}{\rho-\overline{A'O}}=\frac{AA_1}{A_1A'}=\frac{AA_2}{A'A_2}=\frac{\rho+\overline{AO}}{\rho+\overline{A'O}}$$

Next, we solve the proportion

$$\frac{\overline{A0}-\rho}{\rho-\overline{A'O}}=\frac{\rho+\overline{AO}}{\rho+\overline{A'O}}$$

$$\left(\overline{AO}-\rho\right)\left(\rho+\overline{A'O}\right)=\left(\rho-\overline{A'O}\right)\left(\rho+\overline{AO}\right)$$

$$\left(\overline{AO}\right)\rho+\left(\overline{AO}\right)\left(\overline{A'O}\right)-\rho^2-\left(\overline{A'O}\right)\rho=\rho^2+\left(\overline{AO}\right)\rho-\left(\overline{A'O}\right)\rho-\left(\overline{A'O}\right)\left(\overline{AO}\right)$$

$$\left(\overline{AO}\right)\left(\overline{A'O}\right)-\rho^2=\rho^2-\left(\overline{A'O}\right)\left(\overline{AO}\right)$$

$$\left(\overline{AO}\right)\left(\overline{A'O}\right)=\rho^2$$

□

The result defines A' as *the inversion* of the point A through the circle of Apollonius A_1A_2P. [4]

Remarks:

- When $\mu=1$, the locus would be a straight line, the perpendicular bisector of $\overline{AA'}$, that is the line that reflects A into A'. Recall that in this context, a straight line is a circle with infinite radius.
- When $\mu\neq1$ the locus is a circle that inverts A into A'. (Apollonius of Perga)

As a corollary of the above result we will offer the way *inversion* is understood constructively.

Let P be a point in the complete plane. (We include the point at infinity.) Let P' be the inverse image of P in circle C.

- If P is outside of C, to construct P':

A. Construct the two tangent lines of C through P.

B. Draw a line through the points of tangency T_1 and T_2 on the circle.

C. Draw the line $\overline{OP}$. (Recall, O is the center of the circle.)

D. Define $P' = \overline{T_1T_2} \cap \overline{OP}$

- If P is inside of C, to construct P':

A. Extend $\overrightarrow{OP}$ in the plane.

B. Construct the line perpendicular to $\overrightarrow{OP}$ through P.

C. Call the points of intersection of the circle and the line from step B T_1 and T_2.

D. Draw a tangent line of C through T_1 or T_2.

E. P' is the intersection of the tangent line and $\overrightarrow{OP}$

F. $P' = \overline{T_1T_2} \cap \overline{OP}$

- If P is on C, $P' = P$

In general

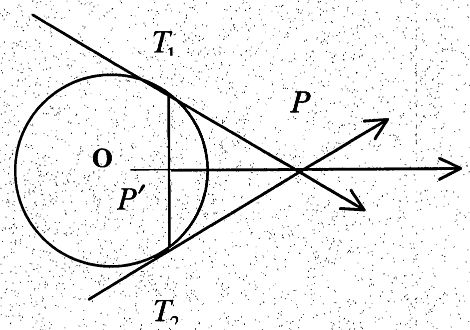


Figure 2.8

These circles are named after the Greek geometer *Apollonius of Perga*, c. 255-170 BC, whose only surviving work is seven out of eight books of a treatise on CONICS, and is also known for improving Aristotle's approximation of π.

(2.11) DEFINITION:

Furthermore, the pencil of all the circles of Apollonius through two fixed points forms the ***hyperbolic*** *(nonintersecting) pencil of circles.*

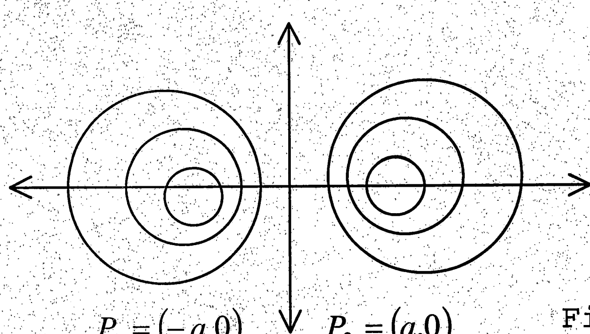


Figure 2.9

The hyperbolic pencil of circles consists of real circles, two different point circles, and, so called "imaginary circles", which will be defined in the following section. No two circles of a given hyperbolic pencil intersect. The investigation of this pencil will continue in Section III. There we will obtain a new, standard representation of all the circles and pencils of circles in the Euclidean plane.

In summary, let P and P' be inverse points in the circle C with origin O. Then:

The pencil of lines through point P' will invert to the elliptic (intersecting) pencil of circles through the given points O and P, including the line $\overline{OPP'}$.

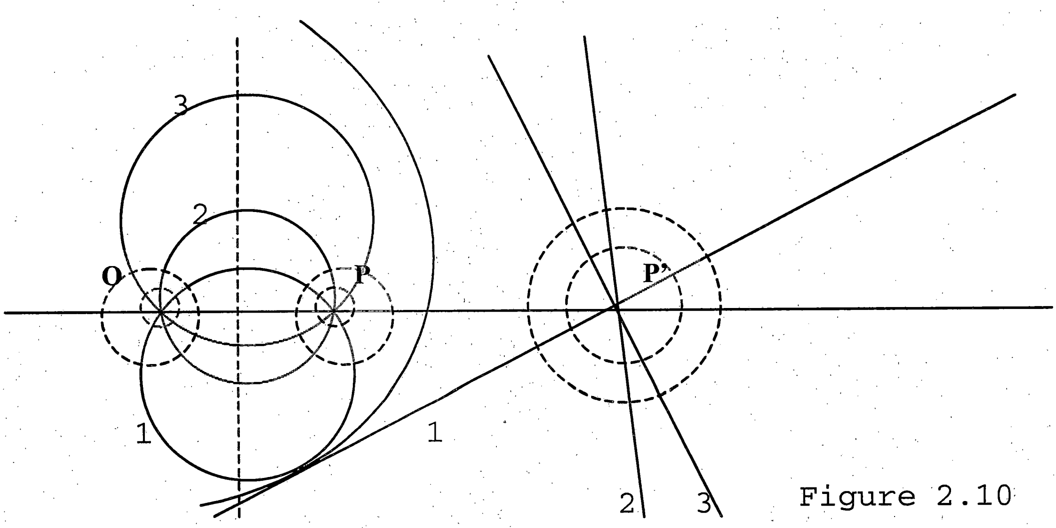


Figure 2.10

The system of concentric circles with center P', which are orthogonal to the lines through point P', will invert to the hyperbolic (nonintersecting) pencil of circles. These circles will not share a point and they will be orthogonal to the elliptic (intersecting) pencil.

In the special case when the center of the circle C coincides with point P, all the circles that touch a fixed line at a fixed point $O = P$ form the parabolic (tangent) pencil of circles, that will invert, in a circle with center O, to all the lines parallel to the given line.

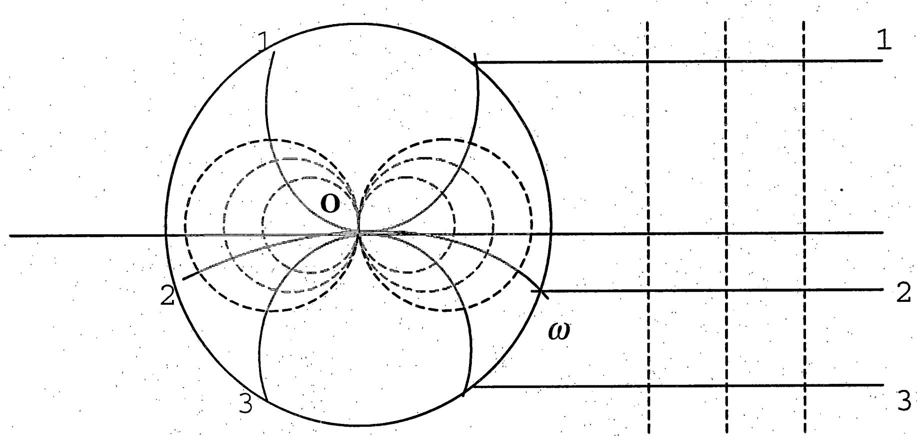


Figure 2.11

III. HERMITIAN REPRESENTATION OF CIRCLES IN THE EUCLIDEAN PLANE

All points $z = x + yi$ of the complex plane, which are of a distance ρ from a fixed point $\gamma = \alpha + \beta i$ form a circle which has a characteristic equation of the form

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2$$

If we rewrite the LHS of this equation as $|z-\gamma|^2 = (z-\gamma)(\bar{z}-\bar{\gamma})$ another form of the circle is obtained, in particular

$$z\bar{z} - \bar{\gamma}z - \gamma\bar{z} + \gamma\bar{\gamma} - \rho^2 = 0$$

The more general equation

$$C(z,\bar{z}) = Az\bar{z} + Bz + C\bar{z} + D = 0\,, \qquad \text{(3.a)}$$

where A and D are real and B and C are complex conjugate numbers, represents the same circle, if

$$B = -A\bar{\gamma}\,, \qquad C = -A\gamma = \bar{B}\,, \qquad D = A(\gamma\bar{\gamma} - \rho^2) \qquad \text{(3.b)}$$

The matrix with entries A, B, $\bar{B}$, and D is therefore a Hermitian matrix

$$C = \begin{pmatrix} A & B \\ \bar{B} & D \end{pmatrix} \qquad \text{(3.c)}$$

(3.1) DEFINITION:

Over the complex numbers, a matrix that is equal to its own *conjugate* transpose is called a *Hermitian matrix.*

The name of such matrix is after the French mathematician Charles Hermite (1822-1901), who worked with Joseph Liouville and Carl Jacobi, and is best known for his proof that e is a transcendental number.

As an example, we use Hermitian matrices to represent the parabolic pencil in which the $y-axis$ is the line l formed by the points- centers of the circles in the pencil (See Figure 2.4).

The origins of the circles are points of the form $(0,k)$ and consequently the radii of the circles in the pencil will be $\rho = k$. Therefore a general equation of a circle in this pencil of circles would be an equation of the form

$$(x-0)^2 + (y-k)^2 = k^2$$

with the circles above the $x-axis$ for $k>0$ and the circles below the $x-axis$ for $k<0$.

Let $A=1$. Then since $C=-A\gamma=\overline{B}$, with $\gamma=0+ik$, it follows that $\overline{B}=-ik$. Consequently the conjugate of $\overline{B}$ must be $B=ik$. Another relationship in (3.b), in particular

$D = A\left(\gamma\overline{\gamma} - \rho^2\right)$ will give $D = 1\left((ik)(-ik) - k^2\right) = -i^2k^2 - k^2 = k^2 - k^2 = 0$.

The Hermitian matrix representing the pencil of circles on Figure 3.1.1 will be of the form

$$C_1 = \begin{pmatrix} 1 & ik \\ -ik & 0 \end{pmatrix} \qquad (3.d)$$

Similarly, the Hermitian matrices representing the pencil of circles whose centers lay on the $x-axis$ (see Figure 2.5) will be of the form

$$C_2 = \begin{pmatrix} 1 & h \\ h & 0 \end{pmatrix} \qquad (3.e)$$

We want now to calculate the inner product.

<u>*(3.2) DEFINITION:*</u>

Two circles are *orthogonal* if the inner product of their Hermitian matrices equals 0. The inner product is defined as the *trace of the inner product of the first matrix and the adjoin of the second matrix,* that is

$$\begin{pmatrix} A_1 & B_1 \\ \overline{B_1} & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ \overline{B_2} & D_2 \end{pmatrix} = A_1D_2 + A_2D_1 - B_1\overline{B}_2 - B_2\overline{B}_1 = tr\begin{pmatrix} A_1 & B_1 \\ \overline{B_1} & D_1 \end{pmatrix}\begin{pmatrix} D_2 & B_2 \\ \overline{B_2} & A_2 \end{pmatrix}.$$

In this case:

$$\begin{pmatrix} 1 & ik \\ -ik & 0 \end{pmatrix}\begin{pmatrix} 1 & h \\ h & 0 \end{pmatrix} = (1)(0) + (1)(0) - ihk - (-ihk) = -ihk + ihk = 0 \qquad (3.f)$$

The result is a real number, which was expected, since for Hermitian matrices the inner product is a real number. The value of the inner product, 0 indicates that these two pencils of circles are orthogonal to each other (see Figure 2.3).

What are the entries of the Hermitian matrices corresponding to the elliptic pencil of circles? We will investigate two separate cases in answering this question:

Case 1: Find the Hermitian matrix corresponding to the unique line circle;

Case 2: Find the Hermitian matrices corresponding to the rest of the circles in the elliptic pencil.

Now:

Case 1: The unique straight-line circle in the elliptic pencil is assigned as the abscissa in the rectangular coordinate system. Therefore the equation of the straight-line circle is $y=0$. Also, in the context of this paper, for a straight-line circle the entry A in the Hermitian matrix

$$\begin{pmatrix} A & B \\ \overline{B} & D \end{pmatrix}$$

has the value of zero. Therefore the general equation of a circle $Az\overline{z}+\overline{B}\overline{z}+Bz+D=0$, for $A=0$ will reduce to

$$\overline{B}\overline{z}+Bz+D=0$$

$$\overline{B}(x-iy)+B(x+iy)+D=0$$

$$\left(B+\overline{B}\right)x+i\left(B-\overline{B}\right)y+D=0$$

In order to get $y=0$ we restrict $B=-\overline{B}$, which gives a pure imaginary (*see footnote* 1) λi with $\lambda>0$ and $\lambda\in R$. We may divide by λ, so the Hermitian matrix takes values

$$\begin{pmatrix} 0 & \pm i \\ \mp i & D \end{pmatrix}.$$

To find the value for D, which will give the desired equation $y=0$, we proceed as

$$\pm i(i+i)y+D=0$$

$$\mp 2iy+D=0$$

Therefore, $y=0$ when $D=0$. Now the Hermitian matrix for the line circle in the elliptic pencil is complete as

$$C_1=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad (3.g)$$

<u>*Case 2:*</u> For the rest of the circles in the elliptic pencil, assume $A=1$. The center of any arbitrary circle is $(0,k)$ or $\gamma=ki$. Then $B=-\gamma\overline{B}=ki$, $\overline{B}=-ki$, and $D=\gamma\overline{\gamma}-\rho^2=k^2-\left(a^2+k^2\right)=-a^2$.

1 For $B=\alpha+i\beta$ and $\overline{B}=\alpha-i\beta$, $B-\overline{B}=(\alpha+i\beta)-(\alpha-i\beta)=2\beta\, i=\lambda\, i$ for $\lambda=2\beta$, a pure imaginary number.

The Hermitian matrix is of a form

$$C_E = \begin{pmatrix} 1 & ki \\ -ki & -a^2 \end{pmatrix} \qquad (3.h)$$

with a determinant $\det = -k^2 - a^2$. As expected, the determinate will always be a negative number.

Observe that, when the distance between the given points P_1 and P_2 is 0, then $P_1 = P_2$ becomes a single given point and the elliptic pencil of circles will be the parabolic pencil of circles (see Figure 2.9).

(3.3) DEFINITION:

The locus of a point P whose distances from two given points P_1 and P_2 are in a constant ratio $1:\mu$, so that $\overline{P_2P} = \mu\overline{P_1P}$ is the circle of Apollonius. Furthermore, the pencil of all the circles of Apollonius determined by two given points forms the hyperbolic pencil of circles.

In a situation where the points P_1 and P_2 have already defined an elliptic pencil, any set of circles through P_1 and P_2 is orthogonal to any circle of Apollonius for P_1 and P_2.

In the special case when $\mu = 1$we get the straight-line circle, the perpendicular bisector of $\overline{P_1P_2}$ or the y-axis of our coordinate system. Let the coordinates of point P be (x, y) and let find the general equation of a circle of Apollonius, corresponding to any value of μ, in this case an arbitrary parameter.

Now, $\overline{P_2P} = \mu\overline{P_1P}$ and by the Distance formula:

$$\sqrt{(x-a)^2 + y^2} = \mu\sqrt{(x+a)^2 + y^2}$$

$$(x-a)^2 + y^2 = \mu^2\left[(x+a)^2 + y^2\right]$$

$$0 = \mu^2\left[(x+a)^2 + y^2\right] - (x-a)^2 - y^2$$

$$0 = \mu^2x^2 + 2\mu^2ax + \mu^2a^2 + \mu^2y^2 - x^2 + 2ax - a^2 - y^2$$

$$0 = x^2(\mu^2 - 1) + 2ax(\mu^2 + 1) + a^2(\mu^2 - 1) + y^2(\mu^2 - 1)$$

$$0 = (\mu^2 - 1)(x^2 + a^2 + y^2) + 2ax(\mu^2 + 1)$$

$$0 = x^2 + a^2 + y^2 + \frac{2a(\mu^2 + 1)}{(\mu^2 - 1)}x$$

$$-a^2 = x^2 + \frac{2a(\mu^2 + 1)}{(\mu^2 - 1)}x + y^2$$

$$-a^2+\frac{a^2(\mu^2+1)^2}{(\mu^2-1)^2}=x^2+\frac{2a(\mu^2+1)}{(\mu^2-1)}x+\frac{a^2(\mu^2+1)^2}{(\mu^2-1)^2}+y^2$$

$$a^2\left(\frac{(\mu^2+1)^2}{(\mu^2-1)^2}-1\right)=\left(x+\frac{a(\mu^2+1)}{(\mu^2-1)}\right)^2+(y-0)^2$$

$$\left(\frac{2a\mu}{\mu^2-1}\right)^2=\left(x+\frac{a(\mu^2+1)}{(\mu^2-1)}\right)^2+(y-0)^2 \qquad (3.i)$$

The resulting equation is an equation of a circle with its center

$$\gamma=\left(\frac{a(\mu^2+1)}{(\mu^2-1)},0\right) \qquad (3.j)$$

with radius

$$\rho=\left(\frac{2a\mu}{\mu^2-1}\right) \qquad (3.k)$$

Indeed, we expected the centers of the circles in the pencil to be on the horizontal axis.

Let's determine the corresponding Hermitian matrix.

Let $A=1$. Then from $C=-A\gamma=\overline{B}$, with $\gamma=\left(\frac{a(\mu^2+1)}{(\mu^2-1)},0\right)$, it follows that

$$\overline{B}=-\frac{a(\mu^2+1)}{(\mu^2-1)}=B\ .$$

Knowing that $D = A\left(\gamma\overline{\gamma} - \rho^2\right)$, we will derive:

$$D = 1\left(\left(-\frac{a\left(\mu^2+1\right)}{\left(\mu^2-1\right)}\right)^2 - \left(\frac{2a\mu}{\mu^2-1}\right)^2\right)$$

$$\left(\frac{a^2\left(\mu^2+1\right)^2 - (2a\mu)^2}{\left(1-\mu^2\right)^2}\right)$$

$$\left(\frac{a^2\left(\mu^2+1+2a\mu\right)\left(\mu^2+1-2a\mu\right)}{\left(1-\mu^2\right)^2}\right)$$

$$\left(\frac{a^2\left(\mu^2+2a\mu+1\right)\left(\mu^2-2a\mu+1\right)}{\left(1-\mu^2\right)^2}\right)$$

$$\left(\frac{a^2(\mu+1)^2(\mu-1)^2}{\left(1-\mu^2\right)^2}\right)$$

$$\left(\frac{a^2((\mu+1)(\mu-1))^2}{\left(1-\mu^2\right)^2}\right)$$

$$\left(\frac{a^2\left(\mu^2-1\right)^2}{\left(1-\mu^2\right)^2}\right)$$

$$a^2 = D$$

The Hermitian matrices representing the hyperbolic pencil of circles are:

$$C_H = \begin{pmatrix} 1 & -\frac{a\left(1+\mu^2\right)}{\left(1-\mu^2\right)} \\ -\frac{a\left(1+\mu^2\right)}{\left(1-\mu^2\right)} & a^2 \end{pmatrix} \qquad (3.1)$$

IV. STEREOGRAPHIC PROJECTION ONTO THE UNIT SPHERE

So far we have been developing the inverse geometry of the plane. As it turns out, the completed plane is a limited field for developing the geometry of circles because the position of one of its points, the point at infinity, is in fact isolated. (The completion of the plane with a *point at infinity* was crucial for the *inversion* of the whole plane onto itself to be a one-one correspondence.)

The idea in this paper is to translate and possibly classify the constructible points of the Euclidean plane onto the sphere. We chose to explore the geometry of circles and a conformal one-one mapping called the Stereographic projection. In order to achieve our main goal in studying constructible circles on the sphere, we will define a mapping from the extended plane onto the sphere with the following properties:

a) It is a topological mapping of the completed plane onto the sphere, which makes these two surfaces topologically equivalent. (A topological mapping is a mapping that is continuous and has a continuous inverse.) The stereographic projection maps from the plane onto the sphere (and vice versa) real circles to real circles, and imaginary circles to imaginary circles.

b) It is a conformal mapping, which means that it preserves angles. The significance of the latter is that the images of orthogonal circles on the plane, under the stereographic projection, are orthogonal circles on the sphere.

Now we are ready to give a precise analytic definition of *pencil of circles* in terms of Hermitian coordinates' representation.

(4.1) DEFINITION:

A pencil of circles is a subspace of the vector space of Hermitian matrices. The collection of Hermitian matrices is a 4-dimensional vector space, which means that we can form real linear combinations. Thus, a pencil determined by two circles is a vector space of dimension 2.

As an application of this Hermitian structure we will show two problems:

- The orthogonality of Elliptic and Hyperbolic pencils of circles; and
- Inversions through the unique circle with the smallest radius in the elliptic circle.

By Definition 3.2, two circles are orthogonal if the inner product of their Hermitian matrices equals 0.

The Hermitian matrices representing the hyperbolic pencil of circles with centers (3.j) and radii (3.k) with an arbitrary parameter μ are:

$$C_H = \begin{pmatrix} 1 & -\dfrac{a(\mu^2+1)}{(\mu^2-1)} \\ -\dfrac{a(\mu^2+1)}{(\mu^2-1)} & a^2 \end{pmatrix} \qquad (4.a)$$

The Hermitian matrices for the elliptic pencil of circles (excluding the line circle) are of the form

$$C_E = \begin{pmatrix} 1 & ki \\ -ki & -a^2 \end{pmatrix} \qquad (4.b)$$

Calculating the inner product as defined in Definition 3.2, yields to:

$$\begin{pmatrix} 1 & -\dfrac{a(\mu^2+1)}{(\mu^2-1)} \\ -\dfrac{a(\mu^2+1)}{(\mu^2-1)} & a^2 \end{pmatrix} \begin{pmatrix} 1 & ki \\ -ki & -a^2 \end{pmatrix} =$$

$$1(-a^2)+1(a^2)-(-ki)\left(-\frac{a(\mu^2+1)}{(\mu^2-1)}\right)-ki\left(-\frac{a(\mu^2+1)}{(\mu^2-1)}\right)=$$

$$-a^2 + a^2 - \frac{kai\ (\mu^2+1)}{(\mu^2-1)} + \frac{kai\ (\mu^2+1)}{(\mu^2-1)} = 0$$

Hence, we may state the following theorem:

(4.2) THEOREM:

The Elliptic and the Hyperbolic pencils of circles are mutually orthogonal.

The proof that the two parabolic pencil of circles with the centers of the circles respectively on the $x-axis$ and $y-axis$ are orthogonal is similar.

Pencils have interesting properties with respect to inversion. The elliptic pencil of circles has a unique circle with smallest radius. For the purpose of this paper, we may assume, without loss of generality, that the radius of this circle is 1, that is $(a,0)$ in this exercise will be $(1,0)$.

Previously, we derived the general form of the point of inversion through the unit circle, centered at the origin, that is

$$\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

Also, we introduced the general equation for a circle (3.a)

$$P_2 = (a,0) = (1,0)$$

$$C(\vec{z},\overline{\vec{z}}) = A\vec{z}\overline{\vec{z}} + B\vec{z} + \overline{B}\overline{\vec{z}} + D = 0$$

The Hermitian matrix for an arbitrary circle in the elliptic pencil was derived as:

$$C_E = \begin{pmatrix} 1 & ki \\ -ki & -a^2 \end{pmatrix}$$

and the Hermitian matrix for the unit circle as the unique circle with the smallest radius in the elliptic pencil is consequently:

$$C_{E_{uc}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (4.c)$$

since $a = 1$ and $k = 0$, where k is the distance between the origin and the center of the arbitrary circle in the elliptic pencil.

To find the inversion for $k \neq 0$:

$$x^2 + y^2 + ki(x + iy) - ki(x - iy) - 1 = 0$$

$$x^2 + y^2 + kix - ky - kix - ky - 1 = 0$$

$$x^2 + y^2 - 2ky - 1 = 0 \qquad (4.d)$$

(The latter is the general equation of an arbitrary circle in the elliptic pencil.)

$$\left(\frac{x}{x^2+y^2}\right)^2+\left(\frac{y}{x^2+y^2}\right)^2-2k\left(\frac{y}{x^2+y^2}\right)-1=0$$

$$\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}-2ky-\left(x^2+y^2\right)=0$$

$$1-2ky-x^2-y^2=0$$

$$x^2+y^2+2ky-1=0 \qquad (4.e)$$

The opposite sign in the equations (4.d) and (4.e) show that the inversion of an arbitrary circle in the elliptic pencil through the unique circle with the smallest radius is a circle on the other side of the line circle. In fact, *setwise*, an inversion through the unique circle with the smallest radius is equivalent to a reflection through the line circle. Note, the equivalency is up to the same set of points. The reflection through the line circle is a linear transformation, while the inversion through a circle is not a linear transformation.

We conclude that *the elliptic pencil is invariant under inversion through the unique circle with the smallest radius.*

There is a couple of ways one can set the spatial relationship between the plane and the sphere in order to employ the stereographic projection. In this paper we will work with the Euclidean plane intersecting the sphere through the "equator" and we will define the South pole, *S*, of the sphere as the image of the point of infinity of the completed plane. The South pole will be the center of the projection.

One may visually imagine the mapping as a ray of light (a straight line) and the source of this light being at the "hole" of the South pole of the sphere. Thus, any straight line would intersect at a single point both surfaces, the plane and the sphere. For our purposes, we will call the point on the complex plane $\vec{z} = u + iv$ the *pre-image* and the point on the sphere $\vec{z}^* = (u', v', w')$ the *image* of the mapping.

Let's recall that the circles on the sphere are the sets of the points of intersection of a plane with the unit sphere. The general equation of a plane is

$$au' + bv' + cw' = d \qquad (4.f)$$

In the special case of $d = 0$, the plane will be passing through the origin of the sphere and the set of points of intersection would form a great circle on the sphere.

Our next step is to establish the equations of the stereographic projection.

Generally, let (ξ_1,η_1,ζ_1) and (ξ_2,η_2,ζ_2)be arbitrary points in space, represented by the position vectors $\vec{r}_1$ and $\vec{r}_2$, respectively. Then, the vector represents any point on the line determined by these two points $\vec{r}=(\xi,\eta,\zeta)=(1-\lambda)\vec{r}_1+\lambda\vec{r}_2$, where λ is a real parameter, that is

$$\begin{cases}\xi=(1-\lambda)\xi_1+\lambda\xi_2\\ \eta=(1-\lambda)\eta_1+\lambda\eta_2\\ \zeta=(1-\lambda)\zeta_1+\lambda\zeta_2\end{cases} \qquad [3]$$

Applying the relation above to the two points of interest to us, the South pole $S=(0,0,-1)$ of the unit sphere S^2 and an arbitrary point on the competed plane $\vec{z}=(u,v,0)$, the image on the sphere $\vec{z}^*=(u',v',w')$will have the co-ordinates:

$$\begin{cases}u'=(1-\lambda)0+\lambda u\\ v'=(1-\lambda)0+\lambda v\\ w'=(1-\lambda)(-1)+0\end{cases}$$

Simplified,

$$u'=\lambda u\ ;\ v'=\lambda v\ ;\ \text{and}\ w'=\lambda-1 \qquad (4.g)$$

To find the value of λ, we will substitute the expressions of (5.b) into the equation of the unit sphere $u'^2+v'^2+w'^2=1$ as follows:

$$u'^2+v'^2+w'^2=\lambda^2u^2+\lambda^2v^2+\lambda^2-2\lambda+1=\lambda^2\left(u^2+v^2+1\right)-2\lambda+1=1 \qquad (4.h)$$

Solving for λ yields the equation

$$\lambda = \frac{2}{u^2 + v^2 + 1} = \frac{2}{1 + \vec{z}\bar{\vec{z}}}$$

Thus, the coordinates of $\vec{z}^* = (u', v', w')$ will be given by

$$u' + iv' = \frac{2\vec{z}}{1 + \vec{z}\bar{\vec{z}}} \text{ and } w' = \frac{1 - \vec{z}\bar{\vec{z}}}{1 + \vec{z}\bar{\vec{z}}} \qquad (4.i)$$

Explicitly,

$$\vec{z}^* = (u', v', w') = \frac{1}{1 + \vec{z}\bar{\vec{z}}}\left(2u, 2v, 1 - \vec{z}\bar{\vec{z}}\right) \qquad (4.j)$$

In the other direction, to find $\vec{z}$ on the plane for a given $\vec{z}^*$ on the sphere, similarly:

$$\begin{cases} u = (1-\lambda)0 + \lambda u' \\ v = (1-\lambda)0 + \lambda v' \\ 0 = (1-\lambda)(-1) + \lambda w' \end{cases}$$

Simplified,

$$u = \lambda u' \text{ ; } v = \lambda v' \text{ ; and}$$

$$0 = \lambda - 1 + \lambda w' = -1 + \lambda(1 + w') \qquad (4.k)$$

Hence, $\lambda = \dfrac{1}{1 + w'}$ and consequently

$$\vec{z} = \frac{u' + v'i}{1 + w'} \qquad (4.l)$$

Trivially, the unit circle on the Euclidean plane will contain all the points $\vec{z} = u + vi$, such that $\vec{z} = \vec{z}^*$. In other words, the points on the unit circle would be fixed under the stereographic projection. Also, the unit circle on the Euclidean plane is a great circle on the unit sphere.

The Hermitian matrix for the unit circle is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Our goal now is to find the equations of the stereographic projection, which would map the Hermitian matrix representations of the points on the plane to the ordered triples, (u', v', w') representations of the points on the sphere. The Hermitian matrix for a generalized circle on the plane is

$$\begin{pmatrix} A & B \\ \overline{B} & D \end{pmatrix}$$

with coefficients from the equation of a circle presented at the beginning of this paper

$$A\vec{z}\bar{\vec{z}} + B\vec{z} + \overline{B}\bar{\vec{z}} + D = 0.$$

A plane may be defined by an equation $au' + bv' + cw' = d$, where:

- when $d = 0$, it is an equation of a plane through the center of the sphere;
- when $d \neq 0$, it is an equation of a plane parallel to a plane through the center of the sphere;

Another way to define a plane is by its normal vector $\vec{n}=(u',v',w')$ with the restriction of $u'^2+v'^2+w'^2=1$ and

$$\vec{z}=\frac{u'+v'i}{1+w'}.$$

Therefore,

$$A\vec{z}\overline{\vec{z}}+B\vec{z}+\overline{B}\overline{\vec{z}}+D=0$$

$$A\left(\frac{u'+v'i}{1+w'}\right)\left(\frac{u'-v'i}{1+w'}\right)+B\left(\frac{u'+v'i}{1+w'}\right)+\overline{B}\left(\frac{u'-v'i}{1+w'}\right)+D=0$$

$$A\left(\frac{u'^2+v'^2i}{(1+w')^2}\right)+B\left(\frac{u'+v'i}{1+w'}\right)+\overline{B}\left(\frac{u'-v'i}{1+w'}\right)+D=0$$

Since $u'^2+v'^2=1-w'^2$

$$A(1-w')+B(u'+v'i)+\overline{B}(u'-v'i)+D(1+w')=0$$

Distributing and combining like terms yields to

$$u'=B+\overline{B};$$

$$v'=i\left(B-\overline{B}\right) \Rightarrow iv'=\overline{B}-B$$
$$w'=D-A;$$

$$\text{and } d=D+A \qquad (4.\mathrm{m})$$

In $au'+bv'+cw'=d$, d shows the "height of the cut" where the plane and the sphere intersect. In the Hermitian matrix, D is such that $D=-\rho^2$, where ρ is the radius of the circle. Any circle centered at the origin of the unit sphere has a Hermitian representation of

$$\begin{pmatrix} A & 0 \\ 0 & -\rho^2 \end{pmatrix} \tag{4.n}$$

We already characterized a hyperbolic pencil as follows:

A hyperbolic pencil of circles consists of real circles, imaginary circles, and two different point circles.

Two circles of a hyperbolic pencil cannot have a common point (see Figure 2.9). However, this is the general case of circles of Apollonius.

We may also work with the so-called *standard hyperbolic pencil* (also referred to as hyperbolic pencil in standard position), which is a special case of the collection of the Apollonius circles on the completed plane. In this case, only one of the two point circles is finite and the other one is the point circle at infinity. It is the collection of concentric circles with a common center at the origin.

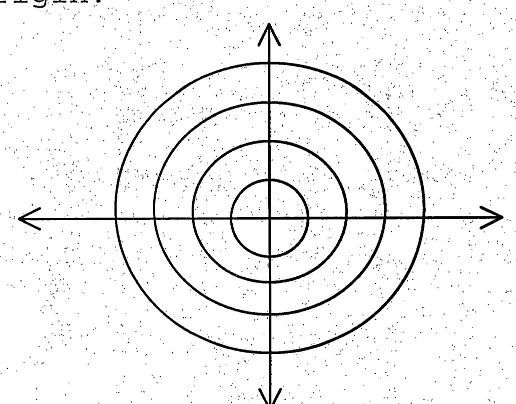
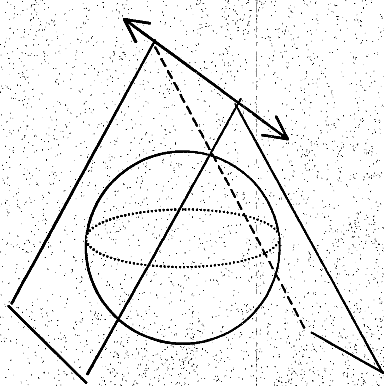


Figure 4.1

The images of this pencil on the sphere, under the stereographic projection, will be circles- sets of points of intersection of "stack" of planes, all parallel to the u,v - plane. The axis of this parallel plane pencil is the line of infinity.

We now want to find the axis corresponding to an arbitrary Apollonius pencil of circles. In order to do this we will observe the images of the two point circles of the hyperbolic pencil onto the sphere.



Apparently, there are two tangent planes to the sphere, whose line of intersection is an line exterior to the sphere, and this line is the axis of the pencil of planes.

Figure 4.2

Let $P_1 = -a_1 + ib_1$ and $P_2 = -a_2 + ib_2$ be the two point circles of the hyperbolic pencil of circles with two distinct Hermitian matrix representations of P_1 and P_2 respectively:

$$\begin{pmatrix} 1 & B_1 \\ \overline{B_1} & D_1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & B_2 \\ \overline{B_2} & D_2 \end{pmatrix}, \text{ where } P_j = -\overline{B}_j = -a_j + ib_j$$

In order to be representations of point circles, the determinants of the Hermitian matrices above must equal 0. Therefore, the restriction on the entries is such that $D-B\overline{B}=D-|B|^2=0$ or $D=|B|^2$, and in order for the two points to be distinct, $D_1 \neq D_2$. Now, the matrix representations could be written in a form

$$\begin{pmatrix} 1 & B_1 \\ \overline{B}_1 & |B_1|^2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & B_2 \\ \overline{B}_2 & |B_2|^2 \end{pmatrix}$$

for the points P_1 and P_2, respectively.

Now, let $B_1 = a_1 + ib_1$ and $B_2 = a_2 + ib_2$. Then $B+\overline{B}=2a$; $B-\overline{B}=2ib$; and $|B|^2 \pm 1 = a^2+b^2 \pm 1$. Substituting these values in (5.g), we have the equations of the images of the two distinct point circles on the sphere as follows:

$$2a_1u' - 2b_1v' + \left(|B_1|^2 - 1\right)w' = |B_1|^2 + 1$$

$$2a_2u' - 2b_2v' + \left(|B_2|^2 - 1\right)w' = |B_2|^2 + 1$$

Note that the plane equations show that the planes will tangent or exterior to the unit sphere.

The points on the line where the two tangent planes containing the images intersect are orthogonal to both normal vectors of the planes.

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2a_1 & -2b_1 & |B_1|^2 - 1 \\ 2a_2 & -2b_2 & |B_2|^2 - 1 \end{vmatrix}$$

$$\vec{d} = 2\left[-b_1\left(|B_2|^2 - 1\right) + b_2\left(|B_1|^2 - 1\right)\right]\vec{i} + 2\left[a_2\left(|B_1|^2 - 1\right) - a_1\left(|B_2|^2 - 1\right)\right]\vec{j} + 4(a_2 b_1 - a_1 b_2)\,\vec{k}$$

Let $P_1 = -a_1 + ib_1$ and $P_2 = -a_2 + ib_2$. Since,

$$P_j^* = \frac{1}{1 + a_j^2 + b_j^2}\left(-2a_j,\ 2b_j,\ 1 - \left(a_j^2 + b_j^2\right)\right), \qquad (4.o)$$

it follows that

$$P_1^* \cdot P_2^* = \frac{1}{\left(1 + a_1^2 + b_1^2\right)\left(1 + a_2^2 + b_2^2\right)}\left(4a_1a_2 + 4b_1b_2 + \left(1 - a_1^2 - b_1^2\right)\left(1 - a_2^2 - b_2^2\right)\right)$$

$$\Rightarrow \frac{4a_1a_2 + 4b_1b_2 + \left(1 - a_1^2 - b_1^2\right)\left(1 - a_2^2 - b_2^2\right)}{\left(1 + a_1^2 + b_1^2\right)\left(1 + a_2^2 + b_2^2\right)} = \cos\theta$$

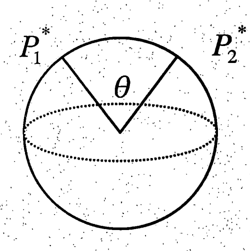


Figure 4.3

P_1^* and P_2^* can be viewed as unit vectors, since they are points on the unit sphere.

$$\frac{1-\alpha}{\alpha}=\cos\theta$$

$$\Rightarrow \alpha\cos\theta = 1-\alpha$$

$$\Rightarrow \alpha(1+\cos\theta)=1$$

$$\Rightarrow \alpha=\frac{1}{1+\cos\theta}$$

Notice that the last result will be undefined in the case where $\cos\theta=-1$. This is a special case and we will explain it later in this section. We will continue our reasoning assuming that $\cos\theta \neq -1$

Combining the last two results yields:

$$\alpha=\frac{\left(1+a_1^2+b_1^2\right)\left(1+a_2^2+b_2^2\right)}{1+4a_1a_2+4b_1b_2+\left(1-a_1^2-b_1^2\right)\left(1-a_2^2-b_2^2\right)}$$

The positional vector

$$\vec{\mathrm{r}}=\alpha\left(P_1^*+P_2^*\right)=\alpha\left(-\left(a_1+a_2\right),\ b_1+b_2,\ 2-a_1^2-a_2^2-b_1^2-b_2^2\right) \tag{4.q}$$

Or $\vec{r}=(\xi,\eta,\zeta)$, that is

$$\begin{cases} \xi = \dfrac{-(a_1+a_2)(1+a_1^2+b_1^2)(1+a_2^2+b_2^2)}{1+4a_1a_2+4b_1b_2+(1-a_1^2-b_1^2)(1-a_2^2-b_2^2)} \\[2em] \eta = \dfrac{(b_1+b_2)(1+a_1^2+b_1^2)(1+a_2^2+b_2^2)}{1+4a_1a_2+4b_1b_2+(1-a_1^2-b_1^2)(1-a_2^2-b_2^2)} \\[2em] \zeta = \dfrac{(2-a_1^2-a_2^2-b_1^2-b_2^2)(1+a_1^2+b_1^2)(1+a_2^2+b_2^2)}{1+4a_1a_2+4b_1b_2+(1-a_1^2-b_1^2)(1-a_2^2-b_2^2)} \end{cases}$$

The point represented by the positional vector

$\vec{r}=(\xi,\eta,\zeta)$

is a point on the axis of the pencil of planes.

The directional vector is

$$\vec{d}=2\left[-b_1\left(a_2^2+b_2^2-1\right)+b_2\left(a_1^2+b_1^2-1\right)\right]\vec{i}$$
$$+2\left[a_2\left(a_1^2+b_1^2-1\right)-a_1\left(a_2^2+b_2^2-1\right)\right]\vec{j}+4(a_2b_1-a_1b_2)\,\vec{k} \tag{4.r}$$

With the location of a point in 3-D and a directional vector, we now can precisely describe a line in 3-D, in particular the axis of a pencil of planes.

Let $P_1=(-\frac{1}{2},0)$ and $P_2=(\frac{1}{2},0)$. Then $a_1=\frac{1}{2}$; $b_1=0$; $a_2=-\frac{1}{2}$; and $b_2=0$. If we substitute these values in the equations above, we have

$$\vec{r} = (\xi, \eta, \zeta) = (0, 0, \tfrac{3}{2}) \text{ and}$$

$$\vec{d} = 0\vec{i} + \tfrac{3}{2}\vec{j} + 0\vec{k} = \tfrac{3}{2}\vec{j}$$

A circle on the sphere is obtained by the intersection of the sphere with the plane. In fact, even planes that do not intersect with the sphere determine "imaginary" circles.

We will finish with few technical results regarding the stereographic image of pencils of circles. The three types of pencils correspond to 3 situations in 3-dimensional pencil of planes (where pencil of planes will be defined as all the planes that contain the line, called axis of the pencil):

A. Elliptic pencil of circles will be determined by a pencil of planes, whose axis is a line secant to the sphere. All the circles will be with a non-zero radius. The points of intersection will be the stereographic projection of the two given points the one that determines the pencil of circles in the plane.

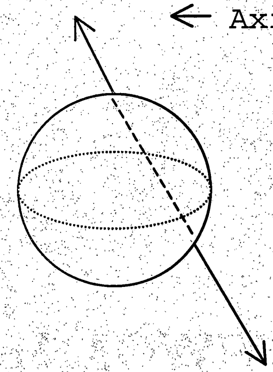


Figure 4.4

B. A pencil of planes, whose axis is a line tangent to the sphere, will determine the parabolic pencil of circles. The image of the unique point circle of the pencil of circles in the plane will occur at the point of tangency.

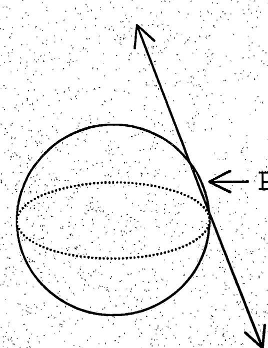
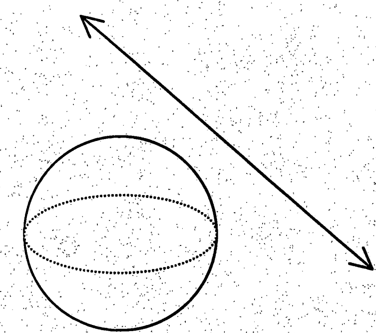


Figure 4.5

C. Hyperbolic (Apollonian) pencil of circles will be determined by a pencil of planes, whose axis is a line exterior to the sphere.

Figure 4.6



I. The two planes tangent to the sphere will determine, under Stereographic projection, the images of *the two point circles in the plane*;

II. All the planes intersecting the sphere will determine the images of the circles with non-zero radii, or *the real circles in the pencil*;

III. All the planes that do not intersect the sphere will determine the images of the circles with a "negative radii" or, so called the *imaginary circles in the pencil.*

How does one, in general, determine the planes in a pencil which produce imaginary circles. We may further elaborate on this using the last concrete example in this section. For $P_1 = (-\frac{1}{2},0)$ and $P_2 = (\frac{1}{2},0)$, the positional vector $\vec{r} = (0,0,\frac{3}{2})$ located a point on the axis of the pencil of planes and the directional vector was $\vec{d} = \frac{3}{2}\vec{j}$.

Therefore, $n_1 x + n_2 y + n_3 z = \frac{3}{2} n_3$, where $\vec{n} = (n_1, n_2, n_3)$, $\vec{d} = \frac{3}{2}\vec{j}$, and $\vec{n} \cdot \vec{d} = 0$ must be a plane in the pencil. Since, $\vec{d} = \frac{3}{2}\vec{j} = (0, \frac{3}{2}, 0)$, $n_1 0 + n_2 \frac{3}{2} + n_3 0 = 0$, which forces $n_2 = 0$. The conclusion is that any vector of the form $\vec{n} = (n_1, 0, n_3)$ is a possible normal vector of this pencil, and any planes of the form $n_1 x + n_3 z = \frac{3}{2} n_3$ for arbitrary values for n_i are planes of this pencil. Back to the concrete case for $P_1 = (-\frac{1}{2}, 0)$ and $P_2 = (\frac{1}{2}, 0)$, using the equations on page 31, we get the images

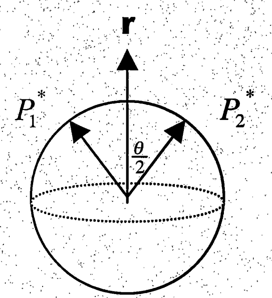
$$P_1^* = \frac{1}{1 + a_1^2 + b_1^2}\left(-2a_1,\ 2b_1,\ 1 - \left(a_1^2 + b_1^2\right)\right) = \tfrac{4}{5}\left(1, 0, \tfrac{3}{4}\right) = \left(\tfrac{4}{5}, 0, \tfrac{3}{5}\right) \text{ and}$$

$$P_2^* = \frac{1}{1 + a_2^2 + b_2^2}\left(-2a_2,\ 2b_2,\ 1 - \left(a_2^2 + b_2^2\right)\right) = \tfrac{4}{5}\left(-1, 0, \tfrac{3}{4}\right) = \left(-\tfrac{4}{5}, 0, \tfrac{3}{5}\right)$$

Thus, planes of the form $\pm\frac{4}{5}x + \frac{3}{5}z = \frac{9}{10}$ or $\pm 8x + 6z = 9$ are planes in the pencil.

It would be even nicer to be able to determine the planes in the Apollonian pencil producing imaginary circles, by the angle θ between the vectors P_j^*.

Figure 4.7



$$\frac{\vec{n}\cdot\vec{r}}{|\vec{n}|\,|\vec{r}|}=\frac{\frac{3}{2}n_3}{\frac{3}{2}\sqrt{n_1^2+n_3^2}}=\frac{n_3}{\sqrt{n_1^2+n_3^2}}<\cos\ \theta\,,$$

where the direction of the inequality follows the fact that $\cos ine$ is a decreasing function. In this particular case, θ is the angle between $P_1^*=\left(\frac{4}{5},0,\frac{3}{5}\right)$ and $P_2^*=\left(-\frac{4}{5},0,\frac{3}{5}\right)$, and can be easily computed as

$$\cos\ \theta=P_1^*\cdot P_2^*=-\tfrac{7}{25}$$

Therefore,

$$\cos\ \frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}=\sqrt{\frac{1+\frac{7}{25}}{2}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

In conclusion, all the planes satisfying the relationship

$$\frac{n_3}{\sqrt{n_1^2+n_3^2}}<\frac{3}{5}$$

will be the planes in the Apollonian pencil producing imaginary circles on the unit sphere.

Although not in this paper, an interesting problem could be posed for the cases where the images of the unit sphere are antipodal points. The tangent planes determining the point circles on the sphere therefore will be parallel to each other, and the axis will be the line at infinity. This will happen when $\cos\theta = -1$.

V. INTRINSIC DEFINITION OF A CONSTRUCTIBLE CIRCLE ON THE UNIT SPHERE

Let's take a tour on the surface of the sphere and adjust to the "laws of nature" working on it. On the sphere, as on the plane, circles have radii, diameters, and chords. The difference is that on the sphere all these are chords of great circles. (Great circles are intersections of the sphere with planes going through its center.) Every circle has exactly two centers- antipodal points on the sphere. Like on the plane, a radius of a circle is one half of its diameter, but unlike of the lengths of radii and diameters on the plane, here, on the sphere, distances are measured in degrees. For some, translation in radian measurements could be more familiar.

The planar straightedge and the planar compass have their counterparts on the sphere. The spherical compass can only draw circles on the sphere with radius measures no longer than 90^o. If a radius greater than 90^o is given, the circle is drawn from the opposite center with radius equal to the supplementary angle.

In the last section of this paper, we will define the constructible circles on the sphere *intrinsically.* We are interested how one can define a circle on the sphere as constructible, using only measurements which can be gathered on the sphere alone, regardless of the preimages in the Euclidean plane. For example, seeing a circle as the intersection of the sphere with a plane in the space, one may calculate the normal and the position vectors determining the plane by taking measurements along the sphere. We will call the normal and the given position vectors that determine the plane, *a set of data* of the plane. Also, angle measurements on the sphere are equivalent to distances. Angles on the sphere can be measured by Lenart's compass, and these are, of course, intrinsic measurements.

In order to achieve this goal, we will start with a non-intrinsic definition of the constructible circles on the sphere.

(5.1) DEFINITION:

A circle on the sphere is constructible if and only if its stereographic projection is constructible on the completed plane.

Our motivation behind this definition is twofold:

- Stereographic projection involves only quadratic rational processes (see Definition 5.2). Thus a point z in the completed plane is constructible if and only if its image, under stereographic projection z^* has consructible coordinates.

- The construction of the circle on the sphere by Lenart instruments is consistent with this definition. We will show this at the end of this section.

(5.2) DEFINITION:

Any calculation from given numbers, involving squaring, square roots, and the four basic operations of a number field, is a *quadratic rational process.*

As an example of a non-intrinsic property we will show that the center of a circle in the plane does not, in general, project to the cup point of the stereographic image of the circle.

Given a circle in the plane defined by $|z-c|=r$, we compute $\vec{z}^*\cdot\vec{c}^*=\cos\theta$, where $(\vec{z}^*\cdot\vec{c}^*)$ is the ordinary dot product of vectors in 3-D and so θ is the angle between them.

Now,

$$|z-c|=r$$

$$\Rightarrow (z-c)(\bar{z}-\bar{c}) = r^2$$

$$\Rightarrow |z|^2 + |c|^2 - (c\bar{z} + \bar{c}z) = r^2$$

$$\Rightarrow |z|^2 + |c|^2 - 2(\operatorname{Re} z \operatorname{Re} c + \operatorname{Im} z \operatorname{Im} c) = r^2$$

$$\Rightarrow |z|^2 + |c|^2 - r^2 = 2(\operatorname{Re} z \operatorname{Re} c + \operatorname{Im} z \operatorname{Im} c) \qquad (5.a)$$

From (4.g) we have the formula for the spherical images of the points on the plane under stereographic projection.

Therefore:

$$\vec{z}^* = \frac{1}{1+|z|^2}\left(2\operatorname{Re} z, 2\operatorname{Im} z, 1-|z|^2\right)$$

and

$$\vec{c}^* = \frac{1}{1+|c|^2}\left(2\operatorname{Re} c, 2\operatorname{Im} c, 1-|c|^2\right)$$

Then

$$\vec{z}^* \cdot \vec{c}^* = \frac{1}{(1+|c|^2)(1+|z|^2)}\left(4\operatorname{Re} z \operatorname{Re} c + 4\operatorname{Im} z \operatorname{Im} c, (1-|z|^2)(1-|c|^2)\right)$$

$$= \frac{1}{(1+|c|^2)(1+|z|^2)}\left(4\operatorname{Re} z \operatorname{Re} c + 4\operatorname{Im} z \operatorname{Im} c + 1 - |z|^2 - |c|^2 + |z|^2|c|^2\right)$$

$$= \frac{1}{(1+|c|^2)(1+|z|^2)}\left(4(\operatorname{Re} z \operatorname{Re} c + \operatorname{Im} z \operatorname{Im} c) + 1 - |z|^2 - |c|^2 + |z|^2|c|^2\right)$$

Substituting the result (5.a) in, we get

$$=\frac{1}{\left(1+|c|^2\right)\left(1+|z|^2\right)}\left(2\ \left(|z|^2+|c|^2-r^2\right)+1-|z|^2-|c|^2+|c|^2|z|^2\right)$$

$$=\frac{1}{\left(1+|c|^2\right)\left(1+|z|^2\right)}\left(1+|z|^2+|c|^2+|z|^2|c|^2-2r^2\right)$$

$$=\frac{1}{\left(1+|c|^2\right)\left(1+|z|^2\right)}\left(\left(1+|z|^2\right)\left(1+|c|^2\right)-2r^2\right)$$

$$=1-\frac{2r^2}{\left(1+|c|^2\right)\left(1+|z|^2\right)} \qquad (5.b)$$

The later is a representation of the spherical distance between the image $\vec{c}*$ of the center of the planar circle and the image $\vec{z}*$ of a point on that circle. Since (5.b) is not a constant expression, $\vec{c}*$ does not necessarily coincide with the cup point- the center of the spherical circle.

We do not want to leave the plane pre-images of the spherical circles before we put a little bit more thought into the constructible circles there. Let's say we have a constructible circle in the Euclidean plane. Then its center and some point on it, say P_1, are constructible points. The diametrical opposite point of P_1, say P_3, must also be a constructible point. Then P_2, the intersection of $\overline{P_1P_3}$ and its perpendicular bisector, would be a constructible point as well. More generally, from Figure 5.1, it is easy to see that $\angle P_1OP_2$, $\angle P_1OP_3$, $\angle P_3OP_2$ are constructible angles, and each of the P_i has coordinates $(\cos\theta, \sin\theta)$ for the appropriate θ, which are constructible entries. We may always translate any constructible circle on the plane, so its center is the origin of the coordinate system, moving through a constructible angle away from a constructible angle. Even if we must scale the circle to get the unit circle, centered at the origin, it will involve multiplication of a constructible distance, which is also constructible.

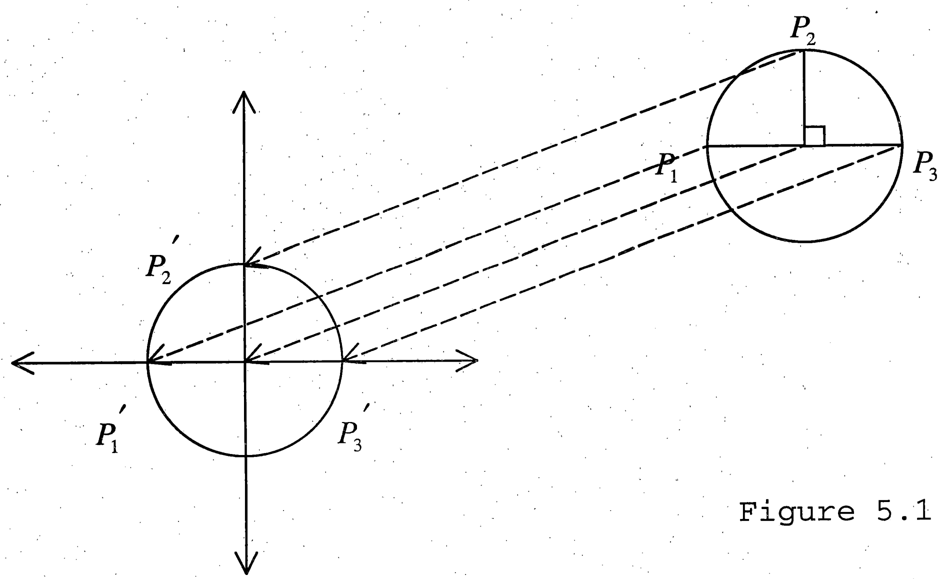


Figure 5.1

It is a logical conclusion, that a constructible circle, defined as such by a constructible point as a center and at least one constructible point in the set of its points, has at least three constructible points in the set of its points.

How may one use this observation for defining a constructible circle on the sphere intrinsically? In other words, how does one define a circle on the sphere as constructible, not knowing what is its preimage on the plane?

Let P_1^*, P_2^*, and P_3^* be the images on the sphere of P_1, P_2, and P_3, respectively. Although they are in different places on the spherical circle now, they are still constructible points. Then, $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ are constructible vectors.

(5.6) DEFINITION:

A vector is constructible if its coordinates are constructible.

NOTE: If a given vector is constructible, then its associated unit vector is also constructible, since division be the length of the vector is a quadratic rational process.

If P_i are constructible:

- Their difference is a constructible distance;
- Their cross product, since it involves
- multiplication and addition only, is

constructible;

- Normalizing their cross product to unit length-

the vector, normal to the plane, determined by the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, involves division by a constructible length.

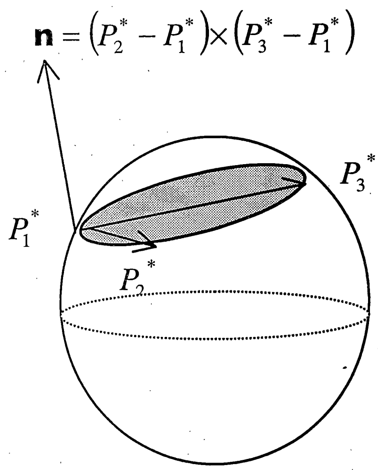


Figure 5.2

A circle on the sphere is constructible if and only if its cup point and its radius are constructible. P_i^* and C^* are constructible and all P_i^* are equidistant from the cup C^*. This distance on the sphere has the angle representation of

$$\cos\ \alpha = P_i^* \cdot C^*$$

(5.7) MAIN THEOREM:

A circle is constructible if and only if it is determined by a plane with constructible data.

(5.8) COROLLARY:

A circle is constructible by Lenart tools if the cup point is constructible and the radius of the circle (angle, arc) is constructible.

(5.9) LEMMA:

A circle on the sphere is constructible only if its Hermitian matrix representation has constructible entries.

(The lemma is conditional only, because the matrix can determine "imaginary" circles,, which we will clear up at the end of this section.)

PROOF: As we have seen in Section III, the entries of the Hermitian matrix, A, B, and D, have been obtained by quadratic rational processes only. Their entries are constructible if the radii of the represented circles are constructible. The data determining the plane (Section IV), in particular the positional vector $\vec{r}$ (4.q) and the normal vector $\vec{n}$ (4.p), involved only quadratic rational processes.

Conversely, it can be shown that a plane corresponds to a constructible circle, provided that it has a normal vector with constructible coordinates, and at least one constructible point on the circle. □

A circle on the sphere is defined as an intersection of the sphere and a plane. We will use the Hermitian matrix to find the plane whose intersection with the sphere is the circle in question. Any plane in 3-D is determined by:

- its normal vector $\vec{n}$; and
- a known position vector $\vec{r}$, which picks up a

point on the plane.

Thus, a constructible circle on the sphere corresponds to a Hermitian matrix with constructible entries. On the other hand, we know that some Hermitian matrices correspond to affine planes that are exterior to the unit sphere. This motivates the following definition:

(5.10) EXTENDED DEFINITION

OF CONSTRUCTIBLE CIRCLE ON THE SPHERE:

A circle on the sphere (real or imaginary) is constructible provided that the plane corresponding to its Hermitian matrix has constructible data.

What are these "imaginary" circles?

Imaginary circles have applications in Hermitian geometry, even though we cannot construct them with Lenart tools. However, the space of Hermitian matrices has a rich geometry of its own where imaginary circles can be given a natural representation. This subject is beyond the scope of this paper, which we hope provides a foundation for such study.

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